

# AN IMPLICIT EULER SCHEME WITH NON-UNIFORM TIME DISCRETIZATION FOR HEAT EQUATIONS WITH MULTIPLICATIVE NOISE

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**ABSTRACT.** We present an algorithm for solving stochastic heat equations, whose key ingredient is a non-uniform time discretization of the driving Brownian motion  $W$ . For this algorithm we derive an error bound in terms of its number of evaluations of one-dimensional components of  $W$ . The rate of convergence depends on the spatial dimension of the heat equation and on the decay of the eigenfunctions of the covariance of  $W$ . According to known lower bounds, our algorithm is optimal, up to a constant, and this optimality cannot be achieved by uniform time discretizations.

## 1. INTRODUCTION

A common technique for the numerical solution of stochastic evolution equations is an Itô-Galerkin approximation, which turns the corresponding infinite-dimensional system of stochastic differential equations (SDEs) into a finite-dimensional one. The latter is then discretized in time and approximately solved by, e.g., an Euler scheme.

More generally, every numerical algorithm for an evolution equation eventually has to discretize the driving stochastic process, which frequently is assumed to be a Brownian motion on an infinite-dimensional Hilbert space, in space and time. The vast majority of algorithms for stochastic evolution equations as well as for SDEs apply a uniform time discretization. This means that a finite number of one-dimensional components of the driving process are evaluated (simulated) at time instances  $\ell/n$  with a common step-size  $1/n$ .

In this paper we present and analyze a non-uniform time discretization for a stochastic heat equation

$$(1) \quad \begin{aligned} dX(t) &= \Delta X(t) dt + B(X(t)) dW(t), \\ X(0) &= \xi \end{aligned}$$

on the Hilbert space  $H = L_2([0, 1]^d)$ . As a key assumption, the system  $(h_i)_{i \in \mathbb{N}^d}$  of eigenfunctions of the trace class covariance  $Q$  of the Brownian motion  $W$  coincides with the system of eigenfunctions of the Laplace operator  $\Delta$ . A finite number of scalar Brownian motions  $\langle W, h_i \rangle$  is selected, and each of them is evaluated with step-size  $1/n_i$  depending on its variance. Based on these data, a properly defined implicit Euler scheme is employed to compute an approximation  $\hat{X}_N^*$  to a finite number of components  $\langle X, h_j \rangle$  of the solution  $X$ . Here  $N$  denotes the total number of evaluations of scalar Brownian motions  $\langle W, h_i \rangle$  used by  $\hat{X}_N^*$ , up to a constant.

Our main result is an upper bound for the error

$$e(\hat{X}_N^*) = \left( E \left( \int_0^1 \|X(t) - \hat{X}_N^*(t)\|^2 dt \right) \right)^{1/2}$$

of  $\hat{X}_N^*$  in terms of  $N$ . The rate of convergence depends on the spatial dimension  $d$  and on the decay of the eigenvalues of the covariance  $Q$ . Assume, for simplicity, that

$$Qh_i = |i|_2^{-\gamma} \cdot h_i$$

for some  $\gamma \in ]d, \infty[ \setminus \{2d\}$ , and put

$$\alpha^*(\gamma, d) = \frac{1}{2} - \frac{(2d - \gamma)_+}{2(d + 2)}.$$

Then

$$e(\hat{X}_N^*) \leq c_1 \cdot N^{-\alpha^*(\gamma, d)}$$

with a constant  $c_1 > 0$  that only depends on  $d$ ,  $\gamma$ ,  $B$ , and  $\xi$ , see Theorem 2.

Actually, this upper bound is best possible, not only for the specific algorithm  $\hat{X}_N^*$  but for any algorithm that uses at most a total of  $N$  evaluations of the scalar Brownian motions  $\langle W, h_i \rangle$ : there exists a constant  $c_2 > 0$  that only depends on  $d$ ,  $\gamma$ ,  $B$ , and  $\xi$  such that

$$e(\hat{X}_N) \geq c_2 \cdot N^{-\alpha^*(\gamma, d)}$$

for any such algorithm. In general, one cannot achieve the optimal rate  $\alpha^*(\gamma, d)$  by any sequence of algorithms that use a uniform discretization. See Müller-Gronbach, Ritter (2006).

In the context of stochastic partial differential equations, implicit (Euler) schemes based on uniform time discretizations are studied, e.g., by Gyöngy (1999), Kloeden, Shott (2001), Hausenblas (2002, 2003), Millet, Morien (2005), Walsh (2005), and Yan (2005). Non-uniform time discretizations are studied for the first time by Müller-Gronbach, Ritter (2006). In the latter paper, non-uniform time discretizations are used for the numerical solution of heat equations with additive noise, i.e.,  $B$  is a function of the time  $t$  but not of the current value  $X(t)$  of the evolution. In this case the solution  $X$  is a Gaussian process and conditional expectations become feasible as a computational tool. This is no longer true for equations with multiplicative noise, as studied in the present paper. Instead, the algorithm introduced in the present paper is a general-purpose algorithm.

Optimality results, as stated here for the algorithm  $\hat{X}_N^*$ , require lower bounds that are valid for all (or at least a broad class) of algorithms. For stochastic evolution equations the first such lower bound is due to Davie, Gaines (2001), who consider a particular case of (1) in spatial dimension  $d = 1$  with a space-time white noise. See Müller-Gronbach, Ritter (2006) for lower bounds for equations (1) in general, with space-time white noise for  $d = 1$  and trace class noise for  $d \geq 1$ .

Our results show the principal significance of non-uniform time discretizations for the numerical solution of stochastic evolution equations. Non-uniform and even sequentially computed time-discretizations are studied, too, for finite-dimensional systems of SDEs. Here, as a rule, those time-discretizations do not improve the order of convergence, but only the asymptotic constants. However, improvements may be substantial on the level of constants, see Cambanis, Hu (1996), Hofmann, Müller-Gronbach, Ritter (2001), and Müller-Gronbach (2002, 2004).

We outline the content of the paper. In Section 2 we formulate the assumptions on the heat equation (1) and briefly discuss existence and uniqueness of a mild solution. Our algorithm is defined in Section 3. Error bounds and optimality properties are stated in Sections 4 and 5, resp., and proofs are given in Section 6.

## 2. ASSUMPTIONS

We study stochastic heat equations (1) on the Hilbert space  $H = L_2(]0, 1[^d)$ . Here  $\xi \in H$  for the initial value, and  $\Delta$  denotes the Laplace operator with Dirichlet boundary conditions on  $H$ . Hence  $\Delta h_{\mathbf{i}} = -\mu_{\mathbf{i}} \cdot h_{\mathbf{i}}$  with

$$h_{\mathbf{i}}(u) = 2^{d/2} \cdot \prod_{\ell=1}^d \sin(i_{\ell} \pi u_{\ell})$$

and

$$\mu_{\mathbf{i}} = \pi^2 \cdot |\mathbf{i}|_2^2,$$

where  $|\mathbf{i}|_2$  is the Euclidean norm of  $\mathbf{i} \in \mathbb{N}^d$ .

Moreover,  $W = (W(t))_{t \in [0,1]}$  denotes a Brownian motion on  $H$ , whose covariance  $Q : H \rightarrow H$  is a trace class operator. Specifically, we assume that  $Qh_{\mathbf{i}} = \lambda_{\mathbf{i}} \cdot h_{\mathbf{i}}$  with

$$\lambda_{\mathbf{i}} = \lambda(|\mathbf{i}|_2)$$

for some non-increasing and regularly varying function

$$\lambda : [1, \infty[ \rightarrow ]0, \infty[$$

of index  $-\gamma$ , where

$$\gamma \in [d, \infty[ \setminus \{2d\}$$

and

$$\int_1^\infty \lambda(r) \cdot r^{d-1} dr < \infty.$$

Note that the latter property always holds if  $\gamma > d$ .

Let  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$  denote the inner product and the norm, respectively, in  $H$ , and consider the Hilbert space  $H_0 = Q^{1/2}(H)$ , equipped with the inner product  $(Q^{1/2}h_1, Q^{1/2}h_2) = \langle h_1, h_2 \rangle$  for  $h_1, h_2 \in H$ . Furthermore, let  $\mathcal{L} = \mathcal{L}_2(H_0, H)$  denote the class of Hilbert-Schmidt operators from  $H_0$  into  $H$ , equipped with the Hilbert-Schmidt norm  $\|\cdot\|_{\mathcal{L}}$ . We assume that the mapping  $B : H \rightarrow \mathcal{L}$  is given by pointwise multiplication and a Nemytskij operator, i.e.,

$$B(x)h = T_g(x) \cdot h$$

for  $x \in H$  and  $h \in H_0$ , where

$$T_g(x) = g \circ x$$

with  $g \in C^1(\mathbb{R})$  such that

$$\|g'\|_{\infty} < \infty.$$

**Remark 1.** We briefly comment on the existence of a mild solution of equation (1) under the above conditions on  $B$ .

Note that  $T_g : H \rightarrow H$ , see Appell, Zabrejko (1990, Thm. 3.1). Furthermore,  $H_0 \subset L_{\infty}(]0, 1[^d)$ , since  $\sup_{\mathbf{i} \in \mathbb{N}^d} \|h_{\mathbf{i}}\|_{\infty} < \infty$ , see Manthey, Zausinger (1999, Lemma 2.2). Hence

$B(x)h \in H$  for every  $h \in H_0$ . Moreover,  $(\lambda_{\mathbf{i}}^{1/2} \cdot h_{\mathbf{i}})_{\mathbf{i} \in \mathbb{N}^d}$  is a complete orthonormal system in  $H_0$  and

$$\sum_{\mathbf{i} \in \mathbb{N}^d} \|B(x)\lambda_{\mathbf{i}}^{1/2}h_{\mathbf{i}}\|^2 \leq \sup_{\mathbf{i} \in \mathbb{N}^d} \|h_{\mathbf{i}}\|_{\infty}^2 \cdot \sum_{\mathbf{i} \in \mathbb{N}^d} \lambda_{\mathbf{i}} \cdot \|T_g(x)\|^2.$$

Consequently,  $B(x) \in \mathcal{L}$  for every  $x \in H$ . Clearly,

$$(2) \quad \|T_g(x) - T_g(y)\| \leq \|g'\|_{\infty} \cdot \|x - y\|$$

for  $x, y \in H$ , which yields

$$(3) \quad \|B(x) - B(y)\|_{\mathcal{L}} \leq K \cdot \|x - y\|$$

with  $K = \sup_{\mathbf{i} \in \mathbb{N}^d} \|h_{\mathbf{i}}\|_{\infty} \cdot \sum_{\mathbf{i} \in \mathbb{N}^d} \lambda_{\mathbf{i}} \cdot \|g'\|_{\infty}$ . Thus  $B : H \rightarrow \mathcal{L}$  is Lipschitz continuous.

Consider the semigroup  $(S(t))_{t \geq 0}$  on  $H$  that is generated by  $\Delta$ , i.e.,

$$S(t)h_{\mathbf{i}} = \exp(-\mu_{\mathbf{i}}t) \cdot h_{\mathbf{i}}.$$

From (3) it follows that there exists a continuous process  $(X(t))_{t \in [0,1]}$  with values in  $H$ , which is adapted to the underlying filtration, such that, for every  $t \in [0, 1]$ ,

$$X(t) = S(t)\xi + \int_0^t S(t-s)B(X(s))dW(s)$$

holds a.s. This process is uniquely determined a.s., and it is called the mild solution of equation (1). Furthermore,

$$(4) \quad \sup_{t \in [0,1]} E\|X(t)\|^2 \leq c_1,$$

where the constant  $c_1 > 0$  only depends on  $d, \xi, \lambda$  and  $g$ . See Da Prato, Zabczyk (1992, Sec. 7.1).

### 3. THE ALGORITHM

We construct and analyze an algorithm that is built from the following ingredients:

- (i) an Itô-Galerkin approximation of the stochastic heat equation,
- (ii) a non-uniform time discretization of the corresponding finite-dimensional Brownian motion,
- (iii) a drift-implicit Euler scheme.

Put

$$\beta_{\mathbf{i}}(t) = \lambda_{\mathbf{i}}^{1/2} \cdot \langle W(t), h_{\mathbf{i}} \rangle$$

for  $\mathbf{i} \in \mathbb{N}^d$  and  $t \in [0, 1]$ . Then  $(\beta_{\mathbf{i}})_{\mathbf{i} \in \mathbb{N}^d}$  is an independent family of standard one-dimensional Brownian motions. Let

$$Y_{\mathbf{j}}(t) = \langle X(t), h_{\mathbf{j}} \rangle$$

for  $t \in [0, 1]$  and  $\mathbf{j} \in \mathbb{N}^d$ . The real-valued processes  $Y_{\mathbf{j}} = (Y_{\mathbf{j}}(t))_{t \in [0,1]}$  satisfy the bi-infinite system

$$\begin{aligned} dY_{\mathbf{j}}(t) &= -\mu_{\mathbf{j}}Y_{\mathbf{j}}(t)dt + \sum_{\mathbf{i} \in \mathbb{N}^d} \lambda_{\mathbf{i}}^{1/2} \cdot \langle B(X(t))h_{\mathbf{i}}, h_{\mathbf{j}} \rangle d\beta_{\mathbf{i}}(t) \\ Y_{\mathbf{j}}(0) &= \langle \xi, h_{\mathbf{j}} \rangle \end{aligned}$$

of stochastic differential equations. For any choice of finite sets  $\mathcal{I}, \mathcal{J} \subseteq \mathbb{N}^d$  an Itô-Galerkin approximation  $\bar{X} = (\bar{X}(t))_{t \in [0,1]}$  to  $X$  is given by

$$\bar{X}(t) = \sum_{j \in \mathcal{J}} \bar{Y}_j(t) \cdot h_j$$

with real-valued processes  $\bar{Y}_j = (\bar{Y}_j(t))_{t \in [0,1]}$  that solve the finite-dimensional system

$$(5) \quad \begin{aligned} d\bar{Y}_j(t) &= -\mu_j \bar{Y}_j(t) dt + \sum_{i \in \mathcal{I}} \lambda_i^{1/2} \cdot \langle B(\bar{X}(t)) h_i, h_j \rangle d\beta_i(t) \\ \bar{Y}_j(0) &= \langle \xi, h_j \rangle. \end{aligned}$$

We apply a drift-implicit Euler scheme to the finite-dimensional system (5). This scheme is based on a non-uniform discretization of the corresponding finite-dimensional Brownian motion, since  $\beta_i$  will be evaluated with step-size  $1/n_i$  depending on  $i \in \mathcal{I}$ . Accordingly, put

$$(6) \quad t_{\ell, i} = \ell/n_i, \quad \ell = 0, \dots, n_i.$$

A good choice of the integers  $n_i \in \mathbb{N}$ , together with sets  $\mathcal{I}$  and  $\mathcal{J}$ , will be presented in Section 4.

In order to understand the construction of this scheme better we first consider a uniform discretization, i.e.,  $t_\ell = t_{\ell, i} = \ell/n$  with a common step-size  $1/n$  for all  $i \in \mathcal{I}$ . In this case the drift-implicit Euler scheme is given by

$$\hat{Y}_j(t_\ell) = \hat{Y}_j(t_{\ell-1}) - \mu_j \hat{Y}_j(t_\ell) \cdot 1/n + \sum_{i \in \mathcal{I}} \lambda_i^{1/2} \cdot \langle B(\hat{X}(t_{\ell-1})) h_i, h_j \rangle \cdot (\beta_i(t_\ell) - \beta_i(t_{\ell-1}))$$

for  $j \in \mathcal{J}$ , where

$$(7) \quad \hat{X}(t) = \sum_{j \in \mathcal{J}} \hat{Y}_j(t) \cdot h_j$$

and

$$(8) \quad \hat{Y}_j(0) = \langle \xi, h_j \rangle.$$

Equivalently,

$$\hat{Y}_j(t_\ell) = \frac{1}{1 + \mu_j/n} \cdot \left( \hat{Y}_j(t_{\ell-1}) + \sum_{i \in \mathcal{I}} \lambda_i^{1/2} \cdot \langle B(\hat{X}(t_{\ell-1})) h_i, h_j \rangle \cdot (\beta_i(t_\ell) - \beta_i(t_{\ell-1})) \right).$$

In general we define

$$0 = \tau_0 < \dots < \tau_M = 1$$

by

$$\{\tau_0, \dots, \tau_M\} = \bigcup_{i \in \mathcal{I}} \{t_{0,i}, \dots, t_{n_i,i}\}.$$

Moreover, we put

$$\mathcal{K}_m = \{i \in \mathcal{I} : \tau_m \in \{t_{0,i}, \dots, t_{n_i,i}\}\}$$

for  $m = 0, \dots, M$ , and we define  $s_{m,i}$  for  $i \in \mathcal{I}$  and  $m = 1, \dots, M$  by

$$s_{m,i} = \max(\{t_{0,i}, \dots, t_{n_i,i}\} \cap [0, \tau_m]).$$

Finally, we use

$$(9) \quad \Gamma_{\mathbf{j}}(t) = \prod_{\nu=1}^M \frac{1}{1 + \mu_{\mathbf{j}} \cdot (t \wedge \tau_{\nu} - t \wedge \tau_{\nu-1})}$$

for approximation of the semigroup generated by  $\Delta$ . Then the drift-implicit Euler scheme is given by (7), (8), and

$$(10) \quad \hat{Y}_{\mathbf{j}}(t) = \frac{\Gamma_{\mathbf{j}}(t)}{\Gamma_{\mathbf{j}}(\tau_{m-1})} \cdot \left( \hat{Y}_{\mathbf{j}}(\tau_{m-1}) + \sum_{\mathbf{i} \in \mathcal{K}_m} \lambda_{\mathbf{i}}^{1/2} \cdot \langle B(\hat{X}(s_{m,\mathbf{i}})) h_{\mathbf{i}}, h_{\mathbf{j}} \rangle \cdot \frac{\Gamma_{\mathbf{j}}(\tau_{m-1})}{\Gamma_{\mathbf{j}}(s_{m,\mathbf{i}})} \cdot (\beta_{\mathbf{i}}(\tau_m) - \beta_{\mathbf{i}}(s_{m,\mathbf{i}})) \right)$$

for  $\mathbf{j} \in \mathcal{J}$ , if

$$t \in ]\tau_{m-1}, \tau_m].$$

Equivalently,

$$(11) \quad \hat{Y}_{\mathbf{j}}(t) = \Gamma_{\mathbf{j}}(t) \cdot \langle \xi, h_{\mathbf{j}} \rangle + \sum_{\mathbf{i} \in \mathcal{I}} \lambda_{\mathbf{i}}^{1/2} \cdot \left( \sum_{t_{\ell,\mathbf{i}} \leq \tau_m} \langle B(\hat{X}(t_{\ell-1,\mathbf{i}})) h_{\mathbf{i}}, h_{\mathbf{j}} \rangle \cdot \frac{\Gamma_{\mathbf{j}}(t)}{\Gamma_{\mathbf{j}}(t_{\ell-1,\mathbf{i}})} \cdot (\beta_{\mathbf{i}}(t_{\ell,\mathbf{i}}) - \beta_{\mathbf{i}}(t_{\ell-1,\mathbf{i}})) \right).$$

For illustration we consider an example with  $\mathcal{I} = \{1, 2\}$ ,  $n_1 = 6$ , and  $n_2 = 4$ . Then, for instance,  $\mathcal{K}_2 = \{2\}$ ,  $\mathcal{K}_3 = \{1\}$ , and  $\mathcal{K}_4 = \{1, 2\}$ . Moreover, for  $t \in ]\tau_2, \tau_3] = ]1/4, 1/3]$  the approximation  $\hat{X}(t)$  is based on the increments  $\beta_1(1/6)$ ,  $\beta_1(1/3) - \beta_1(1/6)$ , and  $\beta_2(1/4)$ , while  $\beta_2(1/2) - \beta_2(1/4)$  is not used at all.

#### 4. ERROR ANALYSIS

Henceforth constants that are hidden in notations like<sup>1</sup>  $\preceq$  and  $\asymp$  may only depend on  $d$ ,  $\xi$ ,  $\lambda$  and  $g$ .

In the sequel we consider the particular choice

$$(12) \quad \begin{aligned} \mathcal{I} &= \{\mathbf{i} \in \mathbb{N}^d : |\mathbf{i}|_2 \leq I\}, \\ \mathcal{J} &= \{\mathbf{j} \in \mathbb{N}^d : |\mathbf{j}|_2 \leq J\} \end{aligned}$$

in the definition of the approximations  $\overline{X}$  and  $\hat{X}$ . Then the error of the Itô-Galerkin approximation  $\overline{X}$  is bounded as follows; see Section 6 for the proof.

**Proposition 1.** *For  $I, J > 0$*

$$E \left( \int_0^1 \|X(t) - \overline{X}(t)\|^2 dt \right) \preceq 1/J^2 + \sum_{|\mathbf{i}|_2 > I} \lambda_{\mathbf{i}}/\mu_{\mathbf{i}}.$$

Moreover, we have the following error bound for the implicit Euler scheme  $\hat{X}$  with an arbitrary discretization (6) specified by  $n \in \mathbb{N}^{\mathcal{T}}$ ; again we refer to Section 6 for the proof.

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<sup>1</sup>Suppose that  $F$  and  $G$  are functions on some set  $A$  with values in  $[0, \infty]$ . By definition,  $F(a) \preceq G(a)$  means  $F(a) \leq c \cdot G(a)$  for all  $a \in A$  with some constant  $c \in ]0, \infty[$ . Furthermore,  $F(a) \asymp G(a)$  means  $F(a) \preceq G(a)$  and  $G(a) \preceq F(a)$ .

**Theorem 1.** For  $I, J > 0$  and  $n \in \mathbb{N}^{\mathcal{I}}$

$$E \left( \int_0^1 \|X(t) - \widehat{X}(t)\|^2 dt \right) \leq 1/J^2 + \sum_{|\mathbf{i}|_2 \leq I} \lambda_{\mathbf{i}}/n_{\mathbf{i}} + \sum_{|\mathbf{i}|_2 > I} \lambda_{\mathbf{i}}/\mu_{\mathbf{i}}.$$

Suppose that  $\widehat{X}$  may use a total of  $N$  evaluations of scalar Brownian motions  $\beta_{\mathbf{i}}$ . Then a proper choice of  $I > 0$  and  $n \in \mathbb{N}^{\mathcal{I}}$  is obtained by minimizing

$$D(I, n) = \sum_{|\mathbf{i}|_2 \leq I} \lambda_{\mathbf{i}}/n_{\mathbf{i}} + \sum_{|\mathbf{i}|_2 > I} \lambda_{\mathbf{i}}/\mu_{\mathbf{i}}$$

under the constraint  $\sum_{|\mathbf{i}|_2 \leq I} n_{\mathbf{i}} \leq N$ . Up to a constant, the corresponding optimization problem is solved as follows.

Recall that, by assumption,

$$(13) \quad \lambda(r) = r^{-\gamma} \cdot L(r)$$

with a slowly varying function  $L : ]1, \infty[ \rightarrow ]0, \infty[$ . Let  $N \in \mathbb{N}$ . We take

$$I = I_N = N^{1/(d+2)}$$

to specify the scalar Brownian motions that are evaluated by the algorithm. For  $\mathbf{i} \in \mathbb{N}^d$  with  $|\mathbf{i}|_2 \leq I$  the Brownian motion  $\beta_{\mathbf{i}}$  is evaluated with step-size  $1/n_{\mathbf{i}}$  where

$$n_{\mathbf{i}} = n_{\mathbf{i},N} = \left\lceil \lambda_{\mathbf{i}}^{1/2} \cdot N^{1-\frac{d-\gamma/2}{d+2}} \cdot (L(N^{1/(d+2)}))^{-1/2} \right\rceil$$

if  $\gamma \in [d, 2d[$  and

$$n_{\mathbf{i}} = n_{\mathbf{i},N} = \left\lceil \lambda_{\mathbf{i}}^{1/2} \cdot N \right\rceil$$

if  $\gamma \in ]2d, \infty[$ . For the total number of evaluations we thus obtain  $\sum_{|\mathbf{i}|_2 \leq I} n_{\mathbf{i}} \asymp N$ . Moreover,

$$\inf \{ D(I, n) : I > 0, n \in \mathbb{N}^{\mathcal{I}} \text{ with } \sum_{|\mathbf{i}|_2 \leq I} n_{\mathbf{i}} \leq N \} \asymp D(I_N, n_N) \asymp e_*^2(N)$$

with

$$e_*(N) = N^{-1/2+\frac{d-\gamma/2}{d+2}} \cdot (L(N^{1/(d+2)}))^{1/2}$$

if  $\gamma \in [d, 2d[$  and

$$e_*(N) = N^{-1/2}$$

if  $\gamma \in ]2d, \infty[$ . See Müller-Gronbach, Ritter (2006).

Finally, we take

$$J = J_N = e_*^{-1}(N).$$

Hereby we have completely specified an algorithm  $\widehat{X} = \widehat{X}_N^*$ .

**Theorem 2.** The error of the algorithm  $\widehat{X}_N^*$  satisfies

$$\left( E \left( \int_0^1 \|X(t) - \widehat{X}_N^*(t)\|^2 dt \right) \right)^{1/2} \leq e_*(N).$$

**Remark 2.** The case of a regularly varying functions  $\lambda$  of index  $-2d$  is not covered by Theorem 2 but may be analyzed in a similar way. Assume, for simplicity, that  $\lambda(r) = r^{-2d}$ . Take  $I_N$  as above, and define

$$n_{\mathbf{i},N} = \lceil \lambda_{\mathbf{i}}^{1/2} \cdot N / \ln N \rceil$$

for  $\mathbf{i} \in \mathbb{N}^d$  with  $|\mathbf{i}|_2 \leq I_N$ . Note that  $\sum_{|\mathbf{i}|_2 \leq I_N} n_{\mathbf{i},N} \asymp N$ . Then

$$\inf\{D(I, n) : I > 0, n \in \mathbb{N}^{\mathcal{I}} \text{ with } \sum_{|\mathbf{i}|_2 \leq I} n_{\mathbf{i}} \leq N\} \asymp D(I_N, n_N) \asymp N^{-1} \cdot (\ln N)^2.$$

Furthermore, take  $J_N = N^{1/2} \cdot (\ln N)^{-1}$ . Due to Theorem 1 the resulting algorithm  $\widehat{X}_N^*$  satisfies

$$\left( E \left( \int_0^1 \|X(t) - \widehat{X}_N^*(t)\|^2 dt \right) \right)^{1/2} \preceq N^{-1/2} \cdot \ln N.$$

**Remark 3.** Consider the implicit Euler scheme  $\widehat{X}$  with a uniform time discretization (6), i.e.,  $t_\ell = t_{\ell,\mathbf{i}} = \ell/n$  for all  $\mathbf{i} \in \mathbb{N}^d$  with  $|\mathbf{i}|_2 \leq I$  and some constant  $n \in \mathbb{N}$ . Assume, for simplicity, that  $\lambda(r) = r^{-\gamma}$  with  $\gamma \in ]d, \infty[$ . By Theorem 1,

$$E \left( \int_0^1 \|X(t) - \widehat{X}(t)\|^2 dt \right) \leq 1/J^2 + d(I, n)$$

with

$$d(I, n) = 1/n \cdot \sum_{|\mathbf{i}|_2 \leq I} |\mathbf{i}|_2^{-\gamma} + \sum_{|\mathbf{i}|_2 > I} |\mathbf{i}|_2^{-(\gamma+2)}.$$

Minimization of this quantity, up to a constant, subject to the constraint  $n \cdot \#\mathcal{I} \leq N$  leads to

$$\inf\{d(I, n) : I > 0, n \in \mathbb{N} \text{ with } n \cdot \#\mathcal{I} \leq N\} \asymp d(I_N, n_N) \asymp N^{-1+\frac{d}{\gamma+2}}$$

with

$$I = I_N = N^{1/(\gamma+2)}$$

and

$$n = n_N = \lceil N^{(\gamma+2-d)/(\gamma+2)} \rceil.$$

Take

$$J = J_N = N^{1/2 - \frac{d}{2(\gamma+2)}},$$

and let  $\widehat{X}_N^{\text{uni}}$  denote the resulting algorithm. By definition,  $n_N \cdot \#\mathcal{I}_N \asymp N$  for the total number of evaluations of scalar Brownian motions  $\beta_{\mathbf{i}}$  used by  $\widehat{X}_N^{\text{uni}}$ , and Theorem 1 yields

$$(14) \quad \left( E \left( \int_0^1 \|X(t) - \widehat{X}_N^{\text{uni}}(t)\|^2 dt \right) \right)^{1/2} \preceq N^{-1/2+\frac{d}{2(\gamma+2)}}.$$

**Remark 4.** We compare the implicit Euler schemes  $\widehat{X}_N^*$  and  $\widehat{X}_N^{\text{uni}}$ , both of which roughly use  $N$  evaluations of scalar Brownian motions. Assume that  $\lambda(r) = r^{-\gamma}$  with  $\gamma \in ]d, \infty[ \setminus \{2d\}$ , and put

$$\alpha^*(\gamma, d) = \frac{1}{2} - \frac{(2d - \gamma)_+}{2(d + 2)}, \quad \alpha(\gamma, d) = \frac{1}{2} - \frac{d}{2(\gamma + 2)}.$$



From Theorem 2 and Remark 3 we get

$$\left( E \left( \int_0^1 \|X(t) - \widehat{X}_N^*(t)\|^2 dt \right) \right)^{1/2} \preceq N^{-\alpha^*(\gamma, d)}$$

for the non-uniform discretization, and

$$\left( E \left( \int_0^1 \|X(t) - \widehat{X}_N^{\text{uni}}(t)\|^2 dt \right) \right)^{1/2} \preceq N^{-\alpha(\gamma, d)}$$

for the uniform discretization.

We always have

$$\alpha^*(\gamma, d) > \alpha(\gamma, d).$$

In the limit for a low degree of smoothness

$$\lim_{\gamma \rightarrow d+} \alpha^*(\gamma, d) = \lim_{\gamma \rightarrow d+} \alpha(\gamma, d) = 1/(d+2).$$

Conversely, for a high degree of smoothness

$$\lim_{\gamma \rightarrow \infty} \alpha(\gamma, d) = 1/2,$$

while  $\alpha^*(\gamma, d) = 1/2$  already holds if  $\gamma > 2d$ .

## 5. OPTIMALITY

The results from Section 4 provide upper bounds for the error of specific algorithms. In particular, the comparison of the implicit Euler schemes based on uniform and non-uniform time discretizations is in fact a comparison only of the corresponding upper bounds. It is therefore important to know whether these upper bounds are lower bounds for the error as well, and, even more, to raise the following questions:

- (i) Does there exist any algorithm  $\widehat{X}_N$  that uses a total of  $N$  evaluations of scalar Brownian motions  $\beta_{\mathbf{i}}$  and achieves an error significantly smaller than the upper bound  $e_*(N)$  for the algorithm  $\widehat{X}_N^*$ ?
- (ii) Are non-uniform time discretizations superior to uniform ones?

To answer these questions we consider arbitrary methods that evaluate a finite number of Brownian motions  $\beta_{\mathbf{i}}$  at a finite number of points and then produce a curve in  $H$  that is close to the corresponding realization of  $X$ . In general, the selection and evaluation of the scalar Brownian motions  $\beta_{\mathbf{i}}$ , is specified by a finite set

$$\mathcal{I} \subseteq \mathbb{N}^d$$

and nodes

$$0 < t_{1,\mathbf{i}} < \dots < t_{n_{\mathbf{i}},\mathbf{i}} \leq 1$$

for  $\mathbf{i} \in \mathcal{I}$  and  $n_{\mathbf{i}} \in \mathbb{N}$ . Every Brownian motion  $\beta_{\mathbf{i}}$  with  $\mathbf{i} \in \mathcal{I}$  is evaluated at the corresponding nodes  $t_{\ell,\mathbf{i}}$ . The total number of evaluations is given by

$$|n|_1 = \sum_{\mathbf{i} \in \mathcal{I}} n_{\mathbf{i}}.$$

Formally, an approximation  $\widehat{X}$  to  $X$  is given by

$$(15) \quad \widehat{X}(t) = \phi(t, \beta_{\mathbf{i}_1}(t_{1,\mathbf{i}_1}), \dots, \beta_{\mathbf{i}_1}(t_{n_{\mathbf{i}_1},\mathbf{i}_1}), \dots, \beta_{\mathbf{i}_k}(t_{1,\mathbf{i}_k}), \dots, \beta_{\mathbf{i}_k}(t_{n_{\mathbf{i}_k},\mathbf{i}_k})),$$

where

$$\phi : [0, 1] \times \mathbb{R}^{|n|_1} \rightarrow H$$

is any measurable mapping and  $\mathcal{I} = \{\mathbf{i}_1, \dots, \mathbf{i}_k\}$ . Here  $\phi$  may depend in any way on the initial value  $\xi$ , the eigenvalues  $\lambda_{\mathbf{i}}$ , and the function  $g$ , which is used to define the mapping  $B$  in the heat equation (1). The error of  $\hat{X}$  is defined by

$$e(\hat{X}) = \left( E \left( \int_0^1 \|X(t) - \hat{X}(t)\|^2 dt \right) \right)^{1/2},$$

cf. Theorems 1 and 2 and the subsequent Remarks.

Let  $\mathfrak{X}_N$  denote the class of all algorithms (15) that use a total of  $N$  evaluations of the scalar Brownian motions  $\beta_{\mathbf{i}}$ , i.e.,  $|n|_1 = N$ . We wish to minimize the error in this class, and hence we study the  $N$ th minimal error

$$e(N) = \inf_{\hat{X} \in \mathfrak{X}_N} e(\hat{X}).$$

In particular, our algorithm  $\hat{X}_N^*$  is of the form (15), and its total number of evaluations of scalar Brownian motions is roughly given by  $N$ .

We obtain a negative answer to Question (i).

**Theorem 3.** *The sequence of algorithms  $\hat{X}_N^*$  is asymptotically optimal, i.e.,*

$$e(\hat{X}_N^*) \asymp e(N),$$

and

$$e(N) \asymp e_*(N).$$

*Proof.* In view of Theorem 2 it remains to show that  $e(N) \succeq e_*(N)$ , and this lower bound is a consequence of a more general result established in Müller-Gronbach, Ritter (2006, Thm. 1).  $\square$

With respect to Question (ii) one needs to study the subclass  $\mathfrak{X}_N^{\text{uni}} \subset \mathfrak{X}_N$  of algorithms that are based on a uniform discretization, i.e.,  $t_{\ell, \mathbf{i}} = \ell/n$  for all  $\mathbf{i} \in \mathcal{I}$  and some constant  $n \in \mathbb{N}$ . The corresponding  $N$ th minimal error in this class is given by

$$e^{\text{uni}}(N) = \inf_{\hat{X} \in \mathfrak{X}_N^{\text{uni}}} e(\hat{X}).$$

**Remark 5.** Consider the specific equation

$$(16) \quad \begin{aligned} dX(t) &= \Delta X(t) dt + dW(t), \\ X(0) &= 0, \end{aligned}$$

i.e.,  $g = 1$  or, equivalently,  $B(x) = \text{id}$ , and assume that  $\lambda(r) = r^{-\gamma}$  with  $\gamma \in ]d, \infty[ \setminus \{2d\}$ . Then

$$(17) \quad e^{\text{uni}}(N) \succeq N^{-1/2 + \frac{d}{2(\gamma+2)}}$$

see Müller-Gronbach, Ritter (2006, Remark 6), so that

$$e(\hat{X}_N^{\text{uni}}) \asymp e^{\text{uni}}(N) \asymp N^{-1/2 + \frac{d}{2(\gamma+2)}}$$

follows from Remark 3.

We thus conclude that the upper bound (14) for the error of the implicit Euler scheme  $\hat{X}_N^{\text{uni}}$  is sharp. Moreover, these algorithms form an asymptotically optimal sequence among all algorithms that use uniform time discretizations. Our comparison of orders of convergence in Remark 4 is therefore a result on minimal errors and clearly shows the superiority of non-uniform time discretizations.

Note that the conclusions from the previous remark only apply to the specific equation (16). We conjecture, however, that the lower bound (17) holds in general, in which case these conclusions hold in general as well.

## 6. PROOFS

We start with regularity properties of the solution  $X$  of equation (1). For the mean-square smoothness of  $X$  we have

$$(18) \quad E\|X(s) - X(t)\|^2 \preceq |t - s| \cdot (1 + \psi(\min(s, t)))$$

for  $s, t \in [0, 1]$  with

$$\psi(t) = \sum_{\mathbf{i} \in \mathbb{N}^d} \mu_{\mathbf{i}} \cdot E(\langle X(t), h_{\mathbf{i}} \rangle^2)$$

satisfying

$$(19) \quad \int_0^1 \psi(t) dt < \infty.$$

See Müller-Gronbach, Ritter (2006, Lemma 1).

Consider the Sobolev space  $W_2^1 = W_2^1([0, 1]^d)$  and its subspace  $W_2^{1,0}$ . Note that  $h_{\mathbf{i}} \in W_2^{1,0}$  and

$$\langle x, h_{\mathbf{i}} \rangle_{W_2^1} = (1 + \mu_{\mathbf{i}}) \cdot \langle x, h_{\mathbf{i}} \rangle$$

for every  $x \in W_2^{1,0}$ . Consequently, the functions  $(1 + \mu_{\mathbf{i}})^{-1/2} \cdot h_{\mathbf{i}}$  form a complete orthonormal system in  $W_2^{1,0}$ , and

$$(20) \quad W_2^{1,0} = \{x \in H : \sum_{\mathbf{i} \in \mathbb{N}^d} \mu_{\mathbf{i}} \cdot \langle x, h_{\mathbf{i}} \rangle^2 < \infty\}$$

as well as

$$(21) \quad \|x\|_{W_2^1}^2 \leq 2 \cdot \sum_{\mathbf{i} \in \mathbb{N}^d} \mu_{\mathbf{i}} \cdot \langle x, h_{\mathbf{i}} \rangle^2$$

for  $x \in W_2^{1,0}$ , which is the Poincaré inequality.

**Lemma 1.** *For Lebesgue-almost every  $t \in [0, 1]$  we have*

$$X(t) \in W_2^{1,0}$$

*with probability one and*

$$\sum_{\mathbf{j} \in \mathbb{N}^d} 1/\mu_{\mathbf{j}} \cdot E\langle T_g(X(t)) \cdot h_{\mathbf{i}}, h_{\mathbf{j}} \rangle^2 \preceq 1/\mu_{\mathbf{i}} \cdot (1 + E\|X(t)\|_{W_2^1}^2)$$

*for every  $\mathbf{i} \in \mathbb{N}^d$ . Moreover,*

$$\int_0^1 E\|X(t)\|_{W_2^1}^2 dt < \infty.$$

*Proof.* Combine (19), (20), and (21) to obtain the first and the last claim.

For the proof of the second claim we note that

$$(22) \quad \|T_g(x)\|_{W_2^1} \preceq 1 + \|x\|_{W_2^1}$$

for  $x \in W_2^1$ , see Appell, Zabrejko (1990, Theorems 9.2 and 9.5). Furthermore, we may assume  $g(0) = 0$  without loss of generality. Then

$$(23) \quad T_g(W_2^{1,0}) \subset W_2^{1,0}$$

is easily verified. In view of (22), (23) and the first statement in the lemma it suffices to show that

$$(24) \quad \sum_{\mathbf{j} \in \mathbb{N}^d} 1/\mu_{\mathbf{j}} \cdot \langle x \cdot h_{\mathbf{i}}, h_{\mathbf{j}} \rangle^2 \preceq 1/\mu_{\mathbf{i}} \cdot \|x\|_{W_2^1}^2$$

for all  $x \in W_2^{1,0}$  and  $\mathbf{i} \in \mathbb{N}^d$ .

To this end fix  $\mathbf{i}, \mathbf{j} \in \mathbb{N}^d$  and  $\ell \in \{1, \dots, d\}$ , and put

$$f_{\mathbf{i}} = 1/(i_{\ell}\pi) \cdot \frac{\partial}{\partial u_{\ell}} h_{\mathbf{i}}, \quad f_{\mathbf{j}} = 1/(j_{\ell}\pi) \cdot \frac{\partial}{\partial u_{\ell}} h_{\mathbf{j}}.$$

Then

$$\begin{aligned} i_{\ell}^2 \cdot \langle x, h_{\mathbf{i}} \cdot h_{\mathbf{j}} \rangle^2 &\asymp \langle x, \frac{\partial}{\partial u_{\ell}} f_{\mathbf{i}} \cdot h_{\mathbf{j}} \rangle^2 \\ &= \left( \langle \frac{\partial}{\partial u_{\ell}} x, f_{\mathbf{i}} \cdot h_{\mathbf{j}} \rangle - j_{\ell}\pi \cdot \langle x, f_{\mathbf{i}} \cdot f_{\mathbf{j}} \rangle \right)^2 \\ &\preceq \langle \frac{\partial}{\partial u_{\ell}} x, f_{\mathbf{i}} \cdot h_{\mathbf{j}} \rangle^2 + j_{\ell}^2 \cdot \langle x, f_{\mathbf{i}} \cdot f_{\mathbf{j}} \rangle^2. \end{aligned}$$

Hereby

$$i_{\ell}^2 \cdot \sum_{\mathbf{j} \in \mathbb{N}^d} 1/\mu_{\mathbf{j}} \cdot \langle x, h_{\mathbf{i}} \cdot h_{\mathbf{j}} \rangle^2 \preceq \left\| \frac{\partial}{\partial u_{\ell}} x \cdot f_{\mathbf{i}} \right\|^2 + \|x \cdot f_{\mathbf{i}}\|^2,$$

and we conclude that

$$\mu_{\mathbf{i}} \cdot \sum_{\mathbf{j} \in \mathbb{N}^d} 1/\mu_{\mathbf{j}} \cdot \langle x \cdot h_{\mathbf{i}}, h_{\mathbf{j}} \rangle^2 \preceq \|x\|_{W_2^1}^2,$$

which yields (24).  $\square$

**6.1. Properties of the Itô-Galerkin approximation.** Let  $P_{\mathcal{I}}$  and  $P_{\mathcal{J}}$  denote the orthogonal projections onto the subspaces  $\text{span}\{h_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$  and  $\text{span}\{h_{\mathbf{j}} : \mathbf{j} \in \mathcal{J}\}$ , respectively, and put

$$\overline{B}(x) = P_{\mathcal{J}} \circ B(x) \circ P_{\mathcal{I}}.$$

Then  $\overline{B} : H \rightarrow \mathcal{L}$  satisfies (3) and  $\overline{X}$  is the mild solution of (1) with  $B$  being replaced by  $\overline{B}$ . Hence

$$(25) \quad \sup_{t \in [0,1]} E \|\overline{X}(t)\|^2 \leq c_1,$$

see (4).

We establish an error bound for piecewise constant interpolation of  $\overline{X}$ .

**Lemma 2.** For  $\mathcal{I}, \mathcal{J} \subset \mathbb{N}^d$  and  $m \in \mathbb{N}$

$$\sum_{\ell=0}^{m-1} \int_{\ell/m}^{(\ell+1)/m} E \|\overline{X}(t) - \overline{X}(\ell/m)\|^2 dt \preceq 1/m.$$

*Proof.* Note that (18) and (19) are valid, too, for  $\overline{X}$  and

$$\overline{\psi}(t) = \sum_{\mathbf{j} \in \mathcal{J}} \mu_{\mathbf{j}} \cdot E(\overline{Y}_{\mathbf{j}}^2(t))$$

instead of  $X$  and  $\psi$ , respectively. For  $\ell \in \{0, \dots, m-1\}$  take  $s_{\ell} \in [\ell/m, (\ell+1)/m]$  with

$$\overline{\psi}(s_{\ell})/m \leq \int_{\ell/m}^{(\ell+1)/m} \overline{\psi}(t) dt.$$

On the first subinterval,

$$\int_0^{1/m} E\|\bar{X}(t) - \bar{X}(0)\|^2 dt \leq 2/m \cdot \sup_{t \in [0,1]} E\|\bar{X}(t)\|^2 \preceq 1/m,$$

see (25). On the subintervals  $[\ell/m, (\ell+1)/m]$  with  $\ell \geq 1$  we proceed as follows. If  $t \in [\ell/m, s_\ell]$ , then

$$\begin{aligned} E\|\bar{X}(t) - \bar{X}(\ell/m)\|^2 &\preceq E\|\bar{X}(t) - \bar{X}(s_{\ell-1})\|^2 + E\|\bar{X}(s_{\ell-1}) - \bar{X}(\ell/m)\|^2 \\ &\preceq 1/m \cdot (1 + \bar{\psi}(s_{\ell-1})) \\ &\preceq 1/m + \int_{(\ell-1)/m}^{\ell/m} \bar{\psi}(s) ds. \end{aligned}$$

If  $t \in [s_\ell, (\ell+1)/m]$ , then

$$\begin{aligned} E\|\bar{X}(t) - \bar{X}(\ell/m)\|^2 &\preceq E\|\bar{X}(t) - \bar{X}(s_\ell)\|^2 + E\|\bar{X}(s_\ell) - \bar{X}(s_{\ell-1})\|^2 + E\|\bar{X}(s_{\ell-1}) - \bar{X}(\ell/m)\|^2 \\ &\preceq 1/m \cdot (1 + \bar{\psi}(s_\ell) + \bar{\psi}(s_{\ell-1})) \\ &\preceq 1/m + \int_{(\ell-1)/m}^{(\ell+1)/m} \bar{\psi}(s) ds. \end{aligned}$$

We conclude that

$$\int_{\ell/m}^{(\ell+1)/m} E\|\bar{X}(t) - \bar{X}(\ell/m)\|^2 dt \preceq 1/m^2 + 1/m \cdot \int_{(\ell-1)/m}^{(\ell+1)/m} \bar{\psi}(s) ds,$$

which completes the proof.  $\square$

*Proof of Proposition 1.* Recall the particular choice (12) of the sets  $\mathcal{I}$  and  $\mathcal{J}$  and let  $c$  denote the right-hand side in Proposition 1. Moreover, let

$$X^{(k)}(t) = \sum_{|\mathbf{j}|_2 \leq J} Y_{\mathbf{j}}^{(k)}(t) \cdot h_{\mathbf{j}}$$

for  $k = 1, 2$  with

$$Y_{\mathbf{j}}^{(1)}(t) = \sum_{|\mathbf{i}|_2 > I} \lambda_{\mathbf{i}}^{1/2} \cdot \int_0^t \exp(-\mu_{\mathbf{j}}(t-s)) \cdot \langle T_g(X(s)) \cdot h_{\mathbf{i}}, h_{\mathbf{j}} \rangle d\beta_{\mathbf{i}}(s)$$

and

$$Y_{\mathbf{j}}^{(2)}(t) = \exp(-\mu_{\mathbf{j}}t) \cdot \langle \xi, h_{\mathbf{j}} \rangle + \sum_{|\mathbf{i}|_2 \leq I} \lambda_{\mathbf{i}}^{1/2} \cdot \int_0^t \exp(-\mu_{\mathbf{j}}(t-s)) \cdot \langle T_g(X(s)) \cdot h_{\mathbf{i}}, h_{\mathbf{j}} \rangle d\beta_{\mathbf{i}}(s).$$

Then

$$X(t) = X^{(1)}(t) + X^{(2)}(t) + \sum_{|\mathbf{j}|_2 > J} Y_{\mathbf{j}}(t) \cdot h_{\mathbf{j}},$$

and consequently

$$\begin{aligned} \int_0^t E\|X(s) - \bar{X}(s)\|^2 ds &\preceq \sum_{|\mathbf{j}|_2 > J} \int_0^1 E(Y_{\mathbf{j}}^2(t)) dt + \int_0^1 E\|X^{(1)}(t)\|^2 dt + \int_0^t E\|X^{(2)}(s) - \bar{X}(s)\|^2 ds. \end{aligned}$$

We have

$$\int_0^1 E(Y_j^{(1)}(t))^2 dt \preceq \sum_{|\mathbf{i}|_2 > I} \lambda_{\mathbf{i}} / \mu_{\mathbf{j}} \cdot \int_0^1 E \langle T_g(X(t)) \cdot h_{\mathbf{i}}, h_{\mathbf{j}} \rangle^2 dt,$$

and therefore

$$\int_0^1 E \|X^{(1)}(t)\|^2 dt \leq \sum_{\mathbf{j} \in \mathbb{N}^d} \int_0^1 E(Y_j^{(1)}(t))^2 dt \preceq \sum_{|\mathbf{i}|_2 > I} \lambda_{\mathbf{i}} / \mu_{\mathbf{i}} \leq c$$

by Lemma 1. Furthermore,

$$\sum_{|\mathbf{j}|_2 \geq J} \int_0^1 E(Y_j^{(2)}(t))^2 dt \preceq 1/J^2 \leq c$$

follows from (2), (4), and  $\sup_{\mathbf{i} \in \mathbb{N}^d} \|h_{\mathbf{i}}\|_{\infty} < \infty$ . Finally, if  $|\mathbf{j}|_2 \leq J$ , then

$$E(Y_j^{(2)}(t) - \bar{Y}_j(t))^2 \leq \sum_{|\mathbf{i}|_2 \leq I} \lambda_{\mathbf{i}} \cdot \int_0^t E \langle T_g(X(s)) - T_g(\bar{X}(s)), h_{\mathbf{i}} h_{\mathbf{j}} \rangle^2 ds,$$

and due to (2)

$$\begin{aligned} E \|X^{(2)}(t) - \bar{X}(t)\|^2 &\preceq \int_0^t E \|T_g(X(s)) - T_g(\bar{X}(s))\|^2 ds \preceq \int_0^t E \|X(s) - \bar{X}(s)\|^2 ds \\ &\preceq 2c + \int_0^t E \|X^{(2)}(s) - \bar{X}(s)\|^2 ds. \end{aligned}$$

Since  $E \langle X^{(1)}(t), X^{(2)}(t) \rangle = 0$ , we get  $E \|X^{(2)}(t)\|^2 \leq E \|X(t)\|^2$ . Use (4) and (25) to conclude that

$$\sup_{t \in [0,1]} E \|X^{(2)}(t) - \bar{X}(t)\|^2 < \infty.$$

It remains to apply Gronwall's Lemma to complete the proof.  $\square$

**6.2. Properties of the implicit Euler scheme.** Recall the definition (9) of  $\Gamma_{\mathbf{j}}$  used for approximation of the semigroup.

**Lemma 3.** *Suppose that  $\mathbf{i} \in \mathcal{I}$  and  $\mathbf{j} \in \mathcal{J}$ . Then, for  $\ell = 0, \dots, n_{\mathbf{i}} - 1$ ,*

$$\int_{t_{\ell, \mathbf{i}}}^1 \frac{\Gamma_{\mathbf{j}}^2(t)}{\Gamma_{\mathbf{j}}^2(t_{\ell, \mathbf{i}})} dt \leq 2/\mu_{\mathbf{j}}$$

as well as

$$\int_{t_{\ell, \mathbf{i}}}^1 \left( \frac{\Gamma_{\mathbf{j}}(t)}{\Gamma_{\mathbf{j}}(t_{\ell, \mathbf{i}})} - \exp(-\mu_{\mathbf{i}}(t - t_{\ell, \mathbf{i}})) \right)^2 dt \preceq 1/n^*,$$

where

$$n^* = \max\{n_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}.$$

Furthermore, for  $0 \leq s \leq t \leq 1$ ,

$$\left| 1 - \frac{\Gamma_{\mathbf{j}}(t)}{\Gamma_{\mathbf{j}}(s)} \right| \leq \min(1, \mu_{\mathbf{j}} \cdot (t - s)).$$

*Proof.* For  $t \in [t_{k,i}, t_{k+1,i}]$  with  $k \geq \ell$

$$\frac{\Gamma_j(t)}{\Gamma_j(t_{\ell,i})} \leq \frac{1}{(1 + \mu_j/n_i)^{k-\ell}} \cdot \frac{1}{1 + \mu_j \cdot (t - t_{k,i})},$$

and therefore

$$\int_{t_{\ell,i}}^1 \frac{\Gamma_j^2(t)}{\Gamma_j^2(t_{\ell,i})} dt \leq \frac{1}{\mu_j + n_i} \cdot \sum_{k=0}^{n_i-1} \frac{1}{(1 + \mu_j/n_i)^{2k}}.$$

Thus, if  $\mu_j/n_i \geq 1$ ,

$$\int_{t_{\ell,i}}^1 \frac{\Gamma_j^2(t)}{\Gamma_j^2(t_{\ell,i})} dt \leq 2/\mu_j,$$

and otherwise

$$\int_{t_{\ell,i}}^1 \frac{\Gamma_j^2(t)}{\Gamma_j^2(t_{\ell,i})} dt \leq \frac{1}{n_i} \cdot \frac{1}{1 - 1/(1 + \mu_j/n_i)^2} \leq 2/\mu_j,$$

too.

For the proof of the second statement put  $k^* = \lceil t_{\ell,i} \cdot n^* \rceil$  and

$$f(t) = \frac{\Gamma_j(t)}{\Gamma_j(t_{\ell,i})} - \exp(-\mu_j(t - t_{\ell,i})).$$

Then  $0 \leq f \leq 1$  and

$$\int_{t_{\ell,i}}^1 f^2(t) dt \leq 1/n^* + \sum_{k=k^*}^{n^*-1} \int_{k/n^*}^{(k+1)/n^*} f^2(t) dt.$$

It remains to show that

$$(26) \quad \sum_{k=k^*}^{n^*-1} \sup_{t \in [k/n^*, (k+1)/n^*]} f^2(t) \leq 1.$$

To this end assume that  $t \in [k/n^*, (k+1)/n^*]$  for some  $k \geq k^*$  in the sequel. Use

$$\frac{\Gamma_j(t)}{\Gamma_j(t_{\ell,i})} \leq \frac{1}{1 + \mu_j \cdot (k^*/n^* - t_{\ell,i})} \cdot \frac{1}{(1 + \mu_j/n^*)^{k-k^*}} \cdot \frac{1}{1 + \mu_j \cdot (t - k/n^*)}$$

to obtain

$$f(t) \leq \left( \frac{1}{(1 + \mu_j/n^*)^{k-k^*}} - \exp(-\mu_j(k - k^*)/n^*) \right) + \exp(-\mu_j(k - k^*)/n^*) \cdot (f_0 + f_1(t))$$

with

$$f_0 = \frac{1}{1 + \mu_j \cdot (k^*/n^* - t_{\ell,i})} - \exp(-\mu_j(k^*/n^* - t_{\ell,i}))$$

and

$$f_1(t) = \frac{1}{1 + \mu_j \cdot (t - k/n^*)} - \exp(-\mu_j(t - k/n^*)).$$

Note that  $1/(1+u) - \exp(-u) \leq \min(1/(1+u), u^2)$  for  $u \geq 0$ . Let  $u = \mu_j/n^*$ . Then

$$\frac{1}{(1 + \mu_j/n^*)^{k-k^*}} - \exp(-\mu_j(k - k^*)/n^*) \leq \frac{1}{(1+u)^{k-k^*-1}} \cdot \min(1/(1+u), u^2)$$

for  $k > k^*$ , and therefore

$$\sum_{k=k^*}^{n^*-1} \left( \frac{1}{(1+u)^{k-k^*}} - \exp(-(k - k^*) \cdot u) \right)^2 \leq \frac{(1+u)^2}{(1+u)^2 - 1} \cdot \min(1/(1+u)^2, u^4) \leq 1.$$

Since  $\max(f_0, f_1(t)) \preceq \min(1, u^2)$ , we have

$$\sum_{k=k^*}^{n^*-1} \exp(-2\mu_j(k - k^*)/n^*) \cdot (f_0 + f_1(t)) \preceq \frac{1}{1 - \exp(-2u)} \cdot \min(1, u^2) \preceq 1,$$

which completes the proof of (26).

For the proof of the third statement let  $s \leq t$ , and assume that  $s \in [\tau_{\kappa-1}, \tau_{\kappa}]$  and  $t \in [\tau_{\nu-1}, \tau_{\nu}]$ . By definition

$$\frac{\Gamma_j(t)}{\Gamma_j(s)} = \frac{1 + \mu_j \cdot (s - \tau_{\kappa-1})}{1 + \mu_j \cdot (\tau_{\kappa} - \tau_{\kappa-1})} \cdot \prod_{\iota=\kappa+1}^{\nu-1} \frac{1}{1 + \mu_j \cdot (\tau_{\iota} - \tau_{\iota-1})} \cdot \frac{1}{1 + \mu_j \cdot (t - \tau_{\nu-1})},$$

which implies

$$\begin{aligned} \left| 1 - \frac{\Gamma_j(t)}{\Gamma_j(s)} \right| &\leq \left| 1 - \frac{1 + \mu_j \cdot (s - \tau_{\kappa-1})}{1 + \mu_j \cdot (\tau_{\kappa} - \tau_{\kappa-1})} \right| \\ &\quad + \sum_{\iota=\kappa+1}^{\nu-1} \left| 1 - \frac{1}{1 + \mu_j \cdot (\tau_{\iota} - \tau_{\iota-1})} \right| + \left| 1 - \frac{1}{1 + \mu_j \cdot (t - \tau_{\nu-1})} \right| \\ &\leq \mu_j \cdot (t - s). \end{aligned}$$

Finally,  $0 < \Gamma_j(t)/\Gamma_j(s) \leq 1$ . □

Put

$$a_{i,j}(t) = E \left( \langle T_g(\hat{X}(t)) \cdot h_i, h_j \rangle^2 \right),$$

and note that

$$(27) \quad \sum_{j \in \mathcal{J}} a_{i,j}(t) \preceq 1 + E \|\hat{X}(t)\|^2$$

due to (2) and  $\sup_{i \in \mathbb{N}^d} \|h_i\|_{\infty} \preceq 1$ .

**Lemma 4.** For  $\mathcal{I}, \mathcal{J} \subset \mathbb{N}^d$  and  $n \in \mathbb{N}^{\mathcal{I}}$

$$\sup_{t \in [0,1]} E \|\hat{X}(t)\|^2 \preceq 1.$$

*Proof.* At first we slightly modify the process  $\hat{X}$  by replacing the Brownian increments  $\beta_i(\tau_m) - \beta_i(s_{m,i})$  in the definition (10) of  $\hat{Y}_j$  by increments  $\beta_i(t) - \beta_i(s_{m,i})$ . More precisely, we consider  $\tilde{X}(t) = \sum_{j \in \mathcal{J}} \tilde{Y}_j(t) \cdot h_j$  with  $\tilde{Y}_j(0) = \langle \xi, h_j \rangle$  and

$$\begin{aligned} \tilde{Y}_j(t) &= \frac{\Gamma_j(t)}{\Gamma_j(\tau_{m-1})} \\ &\quad \cdot \left( \tilde{Y}_j(\tau_{m-1}) + \sum_{i \in \mathcal{K}_m} \lambda_i^{1/2} \cdot \langle B(\tilde{X}(s_{m,i})) h_i, h_j \rangle \cdot \frac{\Gamma_j(\tau_{m-1})}{\Gamma_j(s_{m,i})} \cdot (\beta_i(t) - \beta_i(s_{m,i})) \right) \end{aligned}$$

for  $t \in ]\tau_{m-1}, \tau_m]$ . Note that  $\hat{Y}_j$  and  $\tilde{Y}_j$  as well as  $\hat{X}$  and  $\tilde{X}$  coincide at the points  $\tau_m$ . Moreover, by construction of these processes we have

$$(28) \quad \tilde{Y}_j(\tau_m) \text{ and } \tilde{X}(\tau_m) \text{ are measurable w.r.t. } \sigma(\{\beta_i(t_{\ell,i}) : t_{\ell,i} \leq \tau_m, i \in \mathcal{I}\}).$$

We claim that

$$(29) \quad \sup_{t \in [0,1]} E \|\tilde{X}(t)\|^2 \preceq 1.$$



Assume that  $t \in ]\tau_{m-1}, \tau_m]$  in the following. Observing (28) we obtain

$$\begin{aligned} & E(\tilde{Y}_j(t) - \tilde{Y}_j(\tau_{m-1}))^2 \\ &= \left(1 - \frac{\Gamma_j(t)}{\Gamma_j(\tau_{m-1})}\right)^2 \cdot E(\tilde{Y}_j^2(\tau_{m-1})) + \sum_{i \in \mathcal{K}_m} \lambda_i \cdot a_{i,j}(s_{m,i}) \cdot \frac{\Gamma_j^2(\tau_{m-1})}{\Gamma_j^2(s_{m,i})} \cdot (t - s_{m,i}) \\ &\leq E(\tilde{Y}_j^2(\tau_{m-1})) + \sum_{i \in \mathcal{K}_m} \lambda_i / n_i \cdot a_{i,j}(s_{m,i}). \end{aligned}$$

From (27) we therefore get

$$E\|\tilde{X}(t) - \tilde{X}(\tau_{m-1})\|^2 \preceq 1 + \max_{k=0,\dots,m-1} E\|\tilde{X}(\tau_k)\|^2,$$

and we conclude that

$$f(s) = \sup_{r \in [0,s]} E\|\tilde{X}(r)\|^2$$

is finite for  $s \in [0, 1]$ , since  $E\|\tilde{X}(0)\|^2 = \|\xi\|^2 < \infty$ .

Analogously to (11) we have

$$\begin{aligned} \tilde{Y}_j(t) &= \Gamma_j(t) \cdot \langle \xi, h_j \rangle \\ &+ \sum_{i \in \mathcal{I}} \lambda_i^{1/2} \cdot \left( \sum_{t_{\ell,i} \leq \tau_m} \langle B(\tilde{X}(t_{\ell-1,i})) h_i, h_j \rangle \cdot \frac{\Gamma_j(t)}{\Gamma_j(t_{\ell-1,i})} \cdot (\beta_i(t \wedge t_{\ell,i}) - \beta_i(t_{\ell-1,i})) \right), \end{aligned}$$

which implies

$$E(\tilde{Y}_j^2(t)) = \Gamma_j^2(t) \cdot \langle \xi, h_j \rangle^2 + \sum_{i \in \mathcal{I}} \lambda_i \cdot \left( \sum_{t_{\ell,i} \leq \tau_m} a_{i,j}(t_{\ell-1,i}) \cdot \frac{\Gamma_j^2(t)}{\Gamma_j^2(t_{\ell-1,i})} \cdot (t \wedge t_{\ell,i} - t_{\ell-1,i}) \right)$$

due to the measurability property (28). Use (27) to derive

$$\begin{aligned} E\|\tilde{X}(t)\|^2 &\preceq \|\xi\|^2 + \sum_{i \in \mathcal{I}} \lambda_i \cdot \left( \sum_{t_{\ell,i} \leq \tau_m} (1 + f(t_{\ell-1,i})) \cdot (t \wedge t_{\ell,i} - t_{\ell-1,i}) \right) \\ &\preceq 1 + \int_0^t f(s) ds, \end{aligned}$$

so that (29) follows by means of Gronwall's Lemma.

For the process  $\hat{X}$  we apply (11) and observe (28) again to obtain

$$(30) \quad E(\hat{Y}_j^2(t)) = \Gamma_j^2(t) \cdot \langle \xi, h_j \rangle^2 + \sum_{i \in \mathcal{I}} \lambda_i / n_i \cdot \left( \sum_{t_{\ell,i} \leq \tau_m} a_{i,j}(t_{\ell-1,i}) \cdot \frac{\Gamma_j^2(t)}{\Gamma_j^2(t_{\ell-1,i})} \right).$$

Using (29) we conclude that

$$E\|\hat{X}(t)\|^2 \preceq \|\xi\|^2 + \sum_{i \in \mathcal{I}} \lambda_i \cdot \left( 1 + \max_{\ell=0,\dots,n_i} E\|\tilde{X}(t_{\ell,i})\|^2 \right) \preceq 1.$$

□

We turn to the mean-square regularity of the process  $\hat{X}$ .

**Lemma 5.** For  $\mathcal{I}, \mathcal{J} \subset \mathbb{N}^d$ ,  $n \in \mathbb{N}^{\mathcal{I}}$ , and  $0 \leq s \leq t \leq 1$

$$E\|\widehat{X}(s) - \widehat{X}(t)\|^2 \preceq (t-s) \cdot (1 + \widehat{\psi}(s)) + \sum_{\mathbf{i} \in \mathcal{I}} \lambda_{\mathbf{i}}/n_{\mathbf{i}},$$

where

$$\widehat{\psi}(s) = \sum_{\mathbf{j} \in \mathcal{J}} \mu_{\mathbf{j}} \cdot E(\widehat{Y}_{\mathbf{j}}^2(s)).$$

Moreover,

$$(31) \quad \int_0^1 \widehat{\psi}(s) ds \preceq 1.$$

*Proof.* Since  $s \in ]\tau_{m-1}, \tau_m]$  and  $t_{\ell, \mathbf{i}} \leq \tau_m$  implies  $t_{\ell-1, \mathbf{i}} \leq s$ , we obtain

$$\begin{aligned} \int_0^1 E(\widehat{Y}_{\mathbf{j}}^2(s)) ds &\leq \langle \xi, h_{\mathbf{j}} \rangle^2 \cdot \int_0^1 \Gamma_{\mathbf{j}}^2(s) ds + \sum_{\mathbf{i} \in \mathcal{I}} \lambda_{\mathbf{i}}/n_{\mathbf{i}} \cdot \left( \sum_{\ell=0}^{n_{\mathbf{i}}-1} a_{\mathbf{i}, \mathbf{j}}(t_{\ell, \mathbf{i}}) \cdot \int_{t_{\ell, \mathbf{i}}}^1 \frac{\Gamma_{\mathbf{j}}^2(s)}{\Gamma_{\mathbf{j}}^2(t_{\ell, \mathbf{i}})} ds \right) \\ &\preceq 1/\mu_{\mathbf{j}} \cdot \left( \langle \xi, h_{\mathbf{j}} \rangle^2 + \sum_{\mathbf{i} \in \mathcal{I}} \lambda_{\mathbf{i}}/n_{\mathbf{i}} \cdot \sum_{\ell=0}^{n_{\mathbf{i}}-1} a_{\mathbf{i}, \mathbf{j}}(t_{\ell, \mathbf{i}}) \right) \end{aligned}$$

from (30) and Lemma 3. It follows that

$$\int_0^1 \widehat{\psi}(s) ds \preceq \|\xi\|^2 + \sum_{\mathbf{i} \in \mathcal{I}} \lambda_{\mathbf{i}}/n_{\mathbf{i}} \cdot \sum_{\ell=0}^{n_{\mathbf{i}}-1} (1 + E(\|\widehat{X}(t_{\ell, \mathbf{i}})\|^2)),$$

see (27). Use Lemma 4 to complete the proof of (31).

Assume that  $s < t$  with  $s \in [\tau_{m-1}, \tau_m]$  and  $t \in ]\tau_{\kappa-1}, \tau_{\kappa}]$  for  $m \leq \kappa$ . Then

$$\begin{aligned} E(\widehat{Y}_{\mathbf{j}}(s) - \widehat{Y}_{\mathbf{j}}(t))^2 &= \left( 1 - \frac{\Gamma_{\mathbf{j}}(t)}{\Gamma_{\mathbf{j}}(s)} \right)^2 \cdot E(\widehat{Y}_{\mathbf{j}}^2(s)) + \sum_{\mathbf{i} \in \mathcal{I}} \lambda_{\mathbf{i}}/n_{\mathbf{i}} \cdot \left( \sum_{\ell \in \mathcal{K}_{\mathbf{i}}(s, t)} a_{\mathbf{i}, \mathbf{j}}(t_{\ell-1, \mathbf{i}}) \cdot \frac{\Gamma_{\mathbf{j}}^2(t)}{\Gamma_{\mathbf{j}}^2(t_{\ell-1, \mathbf{i}})} \right), \end{aligned}$$

where

$$\mathcal{K}_{\mathbf{i}}(s, t) = \{\ell \in \{1, \dots, n_{\mathbf{i}}\} : t_{\ell, \mathbf{i}} \in ]\tau_m, \tau_{\kappa}]\}$$

if  $s > \tau_{m-1}$  and

$$\mathcal{K}_{\mathbf{i}}(s, t) = \{\ell \in \{1, \dots, n_{\mathbf{i}}\} : t_{\ell, \mathbf{i}} \in [\tau_m, \tau_{\kappa}]\}$$

if  $s = \tau_{m-1}$ . By Lemma 3

$$E(\widehat{Y}_{\mathbf{j}}(s) - \widehat{Y}_{\mathbf{j}}(t))^2 \preceq \mu_{\mathbf{j}} \cdot (t-s) \cdot E(\widehat{Y}_{\mathbf{j}}^2(s)) + \sum_{\mathbf{i} \in \mathcal{I}} \lambda_{\mathbf{i}}/n_{\mathbf{i}} \cdot \left( \sum_{\ell \in \mathcal{K}_{\mathbf{i}}(s, t)} a_{\mathbf{i}, \mathbf{j}}(t_{\ell-1, \mathbf{i}}) \right).$$

Note that  $\#\mathcal{K}_{\mathbf{i}}(s, t) \leq 1 + n_{\mathbf{i}} \cdot (t-s)$ , and apply (27) together with Lemma 4 to obtain

$$\begin{aligned} E\|\widehat{X}(s) - \widehat{X}(t)\|^2 &\preceq (t-s) \cdot \widehat{\psi}(s) + \sum_{\mathbf{i} \in \mathcal{I}} \lambda_{\mathbf{i}}/n_{\mathbf{i}} \cdot \#\mathcal{K}_{\mathbf{i}}(s, t) \\ &\preceq (t-s) \cdot (1 + \widehat{\psi}(s)) + \sum_{\mathbf{i} \in \mathcal{I}} \lambda_{\mathbf{i}}/n_{\mathbf{i}}, \end{aligned}$$

as claimed.  $\square$

In view of Lemma 4 and Lemma 5 we may proceed as in the proof of Lemma 1 to obtain the following error bound for piecewise constant interpolation of  $\widehat{X}$ .

**Lemma 6.** For  $\mathcal{I}, \mathcal{J} \subset \mathbb{N}^d$ ,  $n \in \mathbb{N}^{\mathcal{I}}$ , and  $\mathbf{i} \in \mathcal{I}$

$$\sum_{\ell=1}^{n_{\mathbf{i}}} \int_{t_{\ell-1,\mathbf{i}}}^{t_{\ell,\mathbf{i}}} E \|\widehat{X}(t) - \widehat{X}(t_{\ell-1,\mathbf{i}})\|^2 dt \preceq 1/n_{\mathbf{i}} + \sum_{\mathbf{i}' \in \mathcal{I}} \lambda_{\mathbf{i}'} / n_{\mathbf{i}'}.$$

*Proof of Theorem 1.* Recall the particular choice (12) of the sets  $\mathcal{I}$  and  $\mathcal{J}$  and consider the corresponding Itô-Galerkin approximation  $\overline{X}$ . Because of Proposition 1 it suffices to show that

$$(32) \quad \int_0^1 E \|\overline{X}(t) - \widehat{X}(t)\|^2 dt \preceq \sum_{|\mathbf{i}|_2 \leq I} \lambda_{\mathbf{i}} / n_{\mathbf{i}}.$$

For  $\nu = 1, 2, 3$  we define

$$U_{\mathbf{j}}^{(\nu)}(t) = \sum_{|\mathbf{i}|_2 \leq I} \lambda_{\mathbf{i}}^{1/2} \cdot \int_0^t \sum_{\ell=0}^{n_{\mathbf{i}}-1} V_{\mathbf{i},\mathbf{j},\ell}^{(\nu)}(s, t) \cdot 1_{[t_{\ell,\mathbf{i}}, t_{\ell+1,\mathbf{i}}]}(s) d\beta_{\mathbf{i}}(s)$$

with

$$\begin{aligned} V_{\mathbf{i},\mathbf{j},\ell}^{(1)}(s, t) &= \exp(-\mu_{\mathbf{j}} \cdot (t - s)) \cdot \langle T_g(\overline{X}(s)) - T_g(\overline{X}(t_{\ell,\mathbf{i}})) \cdot h_{\mathbf{i}}, h_{\mathbf{j}} \rangle, \\ V_{\mathbf{i},\mathbf{j},\ell}^{(2)}(s, t) &= \exp(-\mu_{\mathbf{j}} \cdot (t - s)) \cdot \langle T_g(\overline{X}(t_{\ell,\mathbf{i}})) - T_g(\widehat{X}(t_{\ell,\mathbf{i}})) \cdot h_{\mathbf{i}}, h_{\mathbf{j}} \rangle, \\ V_{\mathbf{i},\mathbf{j},\ell}^{(3)}(s, t) &= \left( \exp(-\mu_{\mathbf{j}} \cdot (t - s)) - \frac{\Gamma_{\mathbf{j}}(t)}{\Gamma_{\mathbf{j}}(t_{\ell,\mathbf{i}})} \right) \cdot \langle T_g(\widehat{X}(t_{\ell,\mathbf{i}})) \cdot h_{\mathbf{i}}, h_{\mathbf{j}} \rangle. \end{aligned}$$

Furthermore, we put

$$U_{\mathbf{j}}^{(4)}(t) = \sum_{\mathbf{i} \in \mathcal{I} \setminus \mathcal{K}_m} \lambda_{\mathbf{i}}^{1/2} \cdot \frac{\Gamma_{\mathbf{j}}(t)}{\Gamma_{\mathbf{j}}(s_{m,\mathbf{i}})} \cdot \langle T_g(\widehat{X}(s_{m,\mathbf{i}})) \cdot h_{\mathbf{i}}, h_{\mathbf{j}} \rangle \cdot (\beta_{\mathbf{i}}(t) - \beta_{\mathbf{i}}(s_{m,\mathbf{i}}))$$

and

$$U_{\mathbf{j}}^{(5)}(t) = \sum_{\mathbf{i} \in \mathcal{K}_m} \lambda_{\mathbf{i}}^{1/2} \cdot \frac{\Gamma_{\mathbf{j}}(t)}{\Gamma_{\mathbf{j}}(s_{m,\mathbf{i}})} \cdot \langle T_g(\widehat{X}(s_{m,\mathbf{i}})) \cdot h_{\mathbf{i}}, h_{\mathbf{j}} \rangle \cdot (\beta_{\mathbf{i}}(\tau_m) - \beta_{\mathbf{i}}(t))$$

if  $t \in ]\tau_{m-1}, \tau_m]$ . Then, by definition,

$$\begin{aligned} \overline{Y}_{\mathbf{j}}(t) - \widehat{Y}_{\mathbf{j}}(t) &= (\exp(-\mu_{\mathbf{j}} t) - \Gamma_{\mathbf{j}}(t)) \cdot \langle \xi, h_{\mathbf{j}} \rangle + U_{\mathbf{j}}^{(1)}(t) + U_{\mathbf{j}}^{(2)}(t) + U_{\mathbf{j}}^{(3)}(t) + U_{\mathbf{j}}^{(4)}(t) - U_{\mathbf{j}}^{(5)}(t). \end{aligned}$$

We separately estimate the terms from the right-hand side of this equation.

Lemma 3 yields

$$(33) \quad \sum_{|\mathbf{j}|_2 \leq J} \left( \langle \xi, h_{\mathbf{j}} \rangle^2 \cdot \int_0^1 (\exp(-\mu_{\mathbf{j}} t) - \Gamma_{\mathbf{j}}(t))^2 dt \right) \preceq 1/n^* \preceq \sum_{|\mathbf{i}|_2 \leq I} \lambda_{\mathbf{i}} / n_{\mathbf{i}}.$$

By (2) and Lemma 2

$$(34) \quad \sum_{|\mathbf{j}|_2 \leq J} E(U_{\mathbf{j}}^{(1)}(t))^2 \preceq \sum_{|\mathbf{i}|_2 \leq I} \lambda_{\mathbf{i}} \cdot \left( \sum_{\ell=0}^{n_{\mathbf{i}}-1} \int_{t_{\ell,\mathbf{i}}}^{t_{\ell+1,\mathbf{i}}} E \|\overline{X}(s) - \overline{X}(t_{\ell,\mathbf{i}})\|^2 ds \right) \preceq \sum_{|\mathbf{i}|_2 \leq I} \lambda_{\mathbf{i}} / n_{\mathbf{i}}.$$

Put

$$f(s) = E \|\overline{X}(s) - \widehat{X}(s)\|^2,$$

which is finite because of (25) and Lemma 4. By (2), Lemma 2, and Lemma 6

(35)

$$\begin{aligned}
& \sum_{|\mathbf{j}|_2 \leq J} E(U_{\mathbf{j}}^{(2)}(t))^2 \\
& \preceq \sum_{|\mathbf{i}|_2 \leq I} \lambda_{\mathbf{i}} \cdot \left( \sum_{\ell=0}^{n_{\mathbf{i}}-1} \int_{t \wedge t_{\ell, \mathbf{i}}}^{t \wedge t_{\ell+1, \mathbf{i}}} \left( E \|\bar{X}(s) - \bar{X}(t_{\ell, \mathbf{i}})\|^2 + E \|\hat{X}(s) - \hat{X}(t_{\ell, \mathbf{i}})\|^2 + f(s) \right) ds \right) \\
& \preceq \sum_{|\mathbf{i}|_2 \leq I} \lambda_{\mathbf{i}}/n_{\mathbf{i}} + \int_0^t f(s) ds.
\end{aligned}$$

Suppose that  $s \in ]t_{\ell, \mathbf{i}}, t_{\ell+1, \mathbf{i}}]$ . Then

$$|\exp(-\mu_{\mathbf{j}}(t-s)) - \exp(-\mu_{\mathbf{j}}(t-t_{\ell, \mathbf{i}}))| \leq \exp(-\mu_{\mathbf{j}}(t-s)) \cdot \mu_{\mathbf{j}}/n_{\mathbf{i}}$$

and therefore

$$\begin{aligned}
& \int_s^1 \left( \exp(-\mu_{\mathbf{j}} \cdot (t-s)) - \frac{\Gamma_{\mathbf{j}}(t)}{\Gamma_{\mathbf{j}}(t_{\ell, \mathbf{i}})} \right)^2 dt \\
& \preceq 1/n_{\mathbf{i}} + \int_{t_{\ell, \mathbf{i}}}^1 \left( \exp(-\mu_{\mathbf{j}} \cdot (t-t_{\ell, \mathbf{i}})) - \frac{\Gamma_{\mathbf{j}}(t)}{\Gamma_{\mathbf{j}}(t_{\ell, \mathbf{i}})} \right)^2 dt \\
& \preceq 1/n_{\mathbf{i}}
\end{aligned}$$

follows from Lemma 3. Hereby

$$\begin{aligned}
& \int_0^1 E(U_{\mathbf{j}}^{(3)}(t))^2 dt \\
& \preceq \sum_{|\mathbf{i}|_2 \leq I} \lambda_{\mathbf{i}} \cdot \left( \int_0^1 \int_0^t \sum_{\ell=0}^{n_{\mathbf{i}}-1} \left( \exp(-\mu_{\mathbf{j}} \cdot (t-s)) - \frac{\Gamma_{\mathbf{j}}(t)}{\Gamma_{\mathbf{j}}(t_{\ell, \mathbf{i}})} \right)^2 \cdot a_{\mathbf{i}, \mathbf{j}}(t_{\ell, \mathbf{i}}) \cdot 1_{]t_{\ell, \mathbf{i}}, t_{\ell+1, \mathbf{i}}]}(s) ds dt \right) \\
& \preceq \sum_{|\mathbf{i}|_2 \leq I} \lambda_{\mathbf{i}}/n_{\mathbf{i}} \cdot \left( \int_0^1 \sum_{\ell=0}^{n_{\mathbf{i}}-1} a_{\mathbf{i}, \mathbf{j}}(t_{\ell, \mathbf{i}}) \cdot 1_{]t_{\ell, \mathbf{i}}, t_{\ell+1, \mathbf{i}}]}(s) ds \right),
\end{aligned}$$

which implies

$$(36) \quad \sum_{|\mathbf{j}|_2 \leq J} \int_0^1 E(U_{\mathbf{j}}^{(3)}(t))^2 dt \preceq \sum_{|\mathbf{i}|_2 \leq I} \lambda_{\mathbf{i}}/n_{\mathbf{i}},$$

see (27) and Lemma 4.

By the same facts,

$$(37) \quad \sum_{|\mathbf{j}|_2 \leq J} E(U_{\mathbf{j}}^{(4)}(t))^2 \preceq \sum_{|\mathbf{i}|_2 \leq I} \lambda_{\mathbf{i}}/n_{\mathbf{i}}$$

and

$$(38) \quad \sum_{|\mathbf{j}|_2 \leq J} E(U_{\mathbf{j}}^{(5)}(t))^2 \preceq \sum_{|\mathbf{i}|_2 \leq I} \lambda_{\mathbf{i}}/n_{\mathbf{i}},$$

if  $t \in ]\tau_{m-1}, \tau_m]$ .

Combining (33)–(38) we obtain

$$\int_0^r f(t) dt \preceq \sum_{|\mathbf{i}|_2 \leq I} \lambda_{\mathbf{i}}/n_{\mathbf{i}} + \int_0^r \int_0^t f(s) ds dt.$$

Finally, apply Gronwall's Lemma to derive  $\int_0^1 f(t) dt \preceq \sum_{|\mathbf{i}|_2 \leq I} \lambda_{\mathbf{i}}/n_{\mathbf{i}}$ , as claimed in (32).  $\square$

#### ACKNOWLEDGEMENTS

This work is supported by the Deutsche Forschungsgemeinschaft (DFG).

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